Euler's Method

We Know $y' = \dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x} \approx \dfrac{y_1 - y_0}{x_1 - x_0}$

We don't need to know y in order to find its values. If we are given $y' = f(x,y)$ and an initial condition $y(x_0) = y_0$ then we can approximate y.

$$y' \approx \frac{y_1 - y_0}{x_1 - x_0} \approx \frac{y_1 - y_0}{h}$$

where $\boxed{\text{"Step size"} \quad h = x_1 - x_0}$

$$\Rightarrow \quad y_1 \approx y_0 + h \cdot y'$$

we are given $y' = f(x,y)$

$$\Rightarrow \quad \boxed{y_1 \approx y_0 + h \cdot f(x_0, y_0)}$$

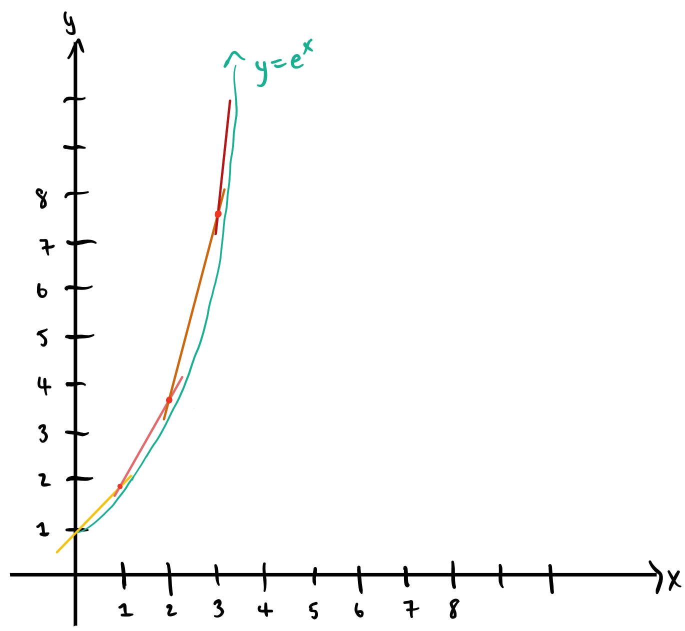
From $x_0 \rightarrow x_1$ (1 step)

ex) Display grafically & tabularly : $\int \frac{dy}{dx} = \int y$ $\quad y(0) = 1 \quad h = 1$

$\downarrow$

$y = e^x$

x	y	$\frac{dy}{dx}$
0	$y_0 =$ 1	1
+h 1	$y_1 \approx 1 + 1(1) = 2$	2
+h 2	$y_2 \approx 2 + 1(2) = 4$	4
+h 3	$y_3 \approx 4 + 1(4) = 8$	8
⋮		



☆ Smaller step size

$\downarrow$

More Accurate

ex) $\frac{dy}{dx} = 3x - 2y$ $\quad y(0) = k \quad h = 1$. Find k such that $y(2) \approx 4.5$

x	y	$\frac{dy}{dx}$
0	k	$-2k$
1	$k + 1\cdot(-2k) = -k$	$3 + 2k$
2	$-k + 1\cdot(3+2k) = 3 + k$	
3	$\vdots$	$\vdots$
$\vdots$		

$3 + k \longrightarrow$ we know $y(2) \approx 4.5$

$\Rightarrow 3 + k = 4.5$

$\Rightarrow \boxed{k = 1.5}$

Protocol:

x	y	$\frac{dy}{dx} = f(x)$
x_1	y_1	$f(x_1, y_1)$
$x_1 + h = x_2$	$y_1 + y'(x_1)\cdot h = y_2$	$f(x_2, y_2)$
$x_2 + h = x_3$	$y_2 + y'(x_2)\cdot h = y_3$	$f(x_3, y_3)$
$\vdots$	$\vdots$	$\vdots$
$x_{i-1} + h = x_i$	$y_{i-1} + y'(x_{i-1})\cdot h = y_i$	$f(x_i, y_i)$

☆ You will be given an initial condition ex. $y(0)=1$ and a step size ex. $h=1$

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ex) $y' = 1 + y$, $y(0) = 1$, $\Delta x = 0.1$ Find y_1, y_2, y_3?

x	y	y'
0	$1 = y_0$	2
0.1	$1 + 2\cdot\frac{1}{10} = 1.2$	$1 + 1.2 = 2.2$
0.2	$1.2 + 2.2\cdot\frac{1}{10} = 1.42$	$1 + 1.42 = 2.42$
0.3	$1.42 + 2.42\cdot\frac{1}{10} = 1.662$	

$y_1 = 1.2$

$y_2 = 1.42$

$y_3 = 1.662$

☆ To find y: multiply the same initial step size Δx $\forall y_i$

ex | $\frac{dy}{dx} = x + y$ let $y = f(x)$ be the solution to this diff. Eq with $f(1) = 2$. Approximate $f(3)$ with 2 steps of equal size.

x	y	$\frac{dy}{dx}$
1	2	3
2	$2 + (1)(3) = 5$	7
3	$5 + (1)(7) = \boxed{12}$	15

$\frac{3-1}{2} = 1$

$f(3) \approx 12$

If we only know:

- the slope of y at any given time ($\frac{dy}{dx}$)
- a starting point ($y(0) = b$)

Then we can approximate y using Euler's method.

Textbook Practice 5-6, 10 (excel)

(5) $\frac{dy}{dx} = 1+y \, , \; y(0)=1 \Rightarrow \int \frac{1}{1+y} dy = \int dx \Rightarrow \ln|1+y| = x + C$

$$1+y = A_0 e^x \Rightarrow y(x) = A_0 e^x - 1$$

$$1 = A_0 - 1 \Rightarrow A_0 = 2$$

$$\boxed{y(x) = 2e^x - 1}$$

(9) $y' = 2y + \sin(x) \, , \; y(0) = 0 \, , \; \Delta x = 0.1$

x	y	y'
0	0	0
0.1	$0 + 0\cdot(0.1) = 0$	$\sin(0.1) = 0.099$
0.2	$0 + (0.09)(0.1) = 0.01$	$2(0.01) + \sin(0.2) = 0.218$
0.3	$0.01 + (0.218)(0.1) = 0.0318$	
$\vdots$	$\vdots$	
1	0.753	

$$\int a^x \, dx = \frac{a^x}{\ln|a|} + C$$

ex) $\int 8^x dx = \frac{8^x}{\ln(8)} + C$

ex) $\int 3^{2x+1} dx = \int 3^{2x} + 3^1 \, dx$

$$= \frac{3^{2x}}{2\ln|3|} + 3x + C$$

$$\frac{dy}{dx} = y \longrightarrow y = Ce^{x}$$